Integrable Wilson Loops from ABJM theory

Nan Bai

Guangxi Normal University

based on the work by N. Drukker and S. Kawamoto, JHEP 07 (2006) 024
and the work in progress

2019.11.23 @ YauC.SEU

1. Correlation function along the loop
2. Integrability of SUSY Wilson loop in SYM
3. Integrability of SUSY Wilson loop in ABJM

1. Correlation function along the loop
2. Integrability of SUSY Wilson loop in SYM
3. Integrability of SUSY Wilson loop in ABJM

Small deformations of the Wilson loops

Consider the Wilson loop operators in $\mathcal{N} = 4$ SYM,

$$W = \frac{1}{N}\mathrm{Tr}\mathcal{P}e^{i\int\left(A_\mu \dot{x}^\mu + i y^i \Phi_i\right)ds} \tag{1}$$

If the path is an infinite straight line or a circle and if it couples to only one of the scalars with the appropriate strength, say $y^i = |\dot{x}|\delta^{i6}$, the WL preserve 1/2 SUSY. Now assume the original loop to be a circle in $(1, 2)$-plane, for a small deformation of the path,

$$x^\mu = x_0^\mu(s) + \epsilon^\mu(s), \quad x_0^\mu = (R\cos s, R\sin s, 0, 0), \quad y^i(s) = |\dot{x}_0|\delta^{i6} + \epsilon^i(s) \tag{2}$$

The WL will deform accordingly,

$$W[x^\mu, y^i] = \left(1 + \int ds \left[\epsilon^\mu \frac{\delta}{\delta x^\mu(s)} + \epsilon^i \frac{\delta}{\delta y^i(s)} + O(\epsilon^2)\right]\right) W_{circle} \tag{3}$$

For our loop deformation, we have

$$W[x^\mu, y^i] = \frac{1}{N}\mathrm{Tr}\mathcal{P}\left[1 + \int ds\,(i\epsilon^\mu(s)\dot{x}_0^\nu(s)F_{\mu\nu} - \epsilon^\mu(s)|\dot{x}_0|D_\mu\Phi_6) - \int ds\epsilon^i(s)|\dot{x}_0|\Phi_i + O(\epsilon^2)\right] e^{i\int(A_\mu\dot{x}_0^\mu + i|\dot{x}_0|\Phi_6)ds} \tag{4}$$

So the deformation of the WL is equivalent to the insertions of the local operators. We define the p-point correlation functions along the loop

$$\boxed{W[\mathcal{O}_p(x_p)\cdots\mathcal{O}_1(x_1)] = \frac{1}{N}\mathrm{Tr}\mathcal{P}\left[\mathcal{O}_p\cdots\mathcal{O}_1 e^{i\int(A_\mu\dot{x}_0^\mu + i|\dot{x}_0|\Phi_6)ds}\right]} \tag{5}$$

which form a gauge invariant observable.

Basic setup

Below are some reviews of the work by Drukker and Kawamoto (2006).
The composite operator is composed of two complex fields

$$Z = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2), \quad X = \frac{1}{\sqrt{2}}(\Phi_3 + i\Phi_4) \tag{6}$$

and we choose the WL to be a straight line along the direction of t,

$$x^0 = t, x^i = 0, i = 1, 2, 3 \tag{7}$$

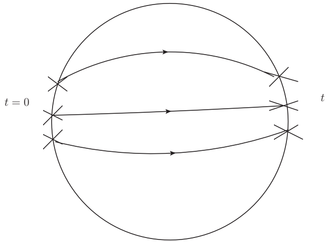
then the 2-pt function of the composite operators along the loop is

$$W[O^\dagger(t)O(0)] = \frac{1}{N}\mathrm{Tr}\mathcal{P}\left[O^\dagger(t)O(0)e^{i\int(A_t + i\Phi_6)dt}\right] \tag{8}$$

We will evaluate this quantity to 1-loop order.

- tree-level: the holonomy will not contribute

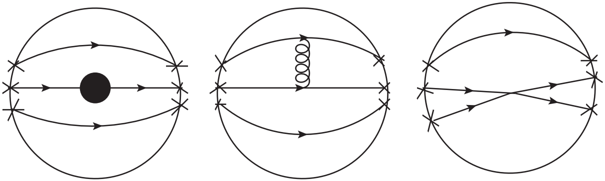
$$W = \langle \frac{1}{N} \mathrm{Tr}\left[O^{\dagger}(t) O(0) \right] \rangle \tag{9}$$



If K is the length of the inserted operator,then

$$\langle W[O^{\dagger}(t)O(0)] \rangle \propto \left(\frac{\lambda}{8\pi^2 t^2} \right)^K \mathbb{I} \tag{10}$$

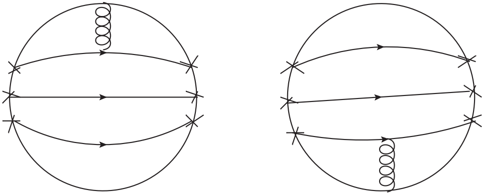
- 1-loop order:
  - bulk terms: these terms come from the interactions between nearest sites within the inserted operators.



The Z-factors of these diagrams are

$$
\begin{aligned}
Z_{\text{self-energy}} &= \mathbb{I} + \frac{\lambda}{8\pi^2} \log \Lambda \mathbb{I}, \\
Z_H &= \mathbb{I} - \frac{\lambda}{16\pi^2} \log \Lambda \mathbb{I}, \\
Z_X &= \mathbb{I} + \frac{\lambda}{16\pi^2} (\mathbb{I} - 2\mathbb{P}) \log \Lambda
\end{aligned}
\tag{11}
$$

- 1-loop order:
  - boundary terms: The interaction between the outermost fields and the WL provides the bdy terms for the open spin chain.



The Z-factor is

$$Z_{\mathsf{bdy}} = \mathbb{I} - \frac{\lambda}{8\pi^2} \log \Lambda \mathbb{I} \tag{12}$$

Integrability in $SU(2)$ sector

- The total 1-loop renormalization factor is

$$Z_{total} = \mathbb{I} + \frac{\lambda}{8\pi^2} \log \Lambda \sum_{l=1}^{K-1} (\mathbb{I} - \mathbb{P}_{l,l+1}) \tag{13}$$

- The ADM is

$$\Gamma = \frac{d \log Z}{d \log \Lambda} \sim \frac{\lambda}{8\pi^2} \sum_{l=1}^{K-1} (\mathbb{I} - \mathbb{P}_{l,l+1}) \tag{14}$$

- doubling trick: the model is equivalent to a regular closed Heisenberg chain of length $2K$ with reflection symmetry.

Doubling trick

Consider two copies of the spin chains with the same spin structure

$$H_1 = \frac{\lambda}{8\pi^2} \sum_{k=1}^{K-1} (\mathbb{I} - \mathbb{P}_{k,k+1}), \quad H_2 = \frac{\lambda}{8\pi^2} \sum_{k=K+1}^{2K} (\mathbb{I} - \mathbb{P}_{k,k+1}) \tag{15}$$

Because the spins at positions k and $2K+1-k$ are the same, the following term will vanish

$$H_3 = \frac{\lambda}{8\pi^2} \left(I - P_{K,K+1} + I - P_{2K,1}\right) \tag{16}$$

Sum up these three terms we get the regular closed Heisenberg spin chain of length $2K$ with reflection symmetry.

Review of ABJM theory

- $\mathcal{N} = 6$ superconformal Chern-Simons matter theory (ABJM theory) was proposed as a $U(N) \times U(N)$ gauge theory to describe a stack of M2 branes at a Z_k orbifold point. In the large N limit, its gravity dual is type IIA string theory on $AdS_4 \times CP^3$.
- ABJM theory has a Lagrangian description

$$\begin{aligned} I &= \frac{k}{4\pi} \int_{\mathbb{R}^{2,1}} \Big(CS(A) - CS(\hat{A}) \Big) \\ &\quad - \mathrm{Tr}(D_\mu Y)^\dagger D^\mu Y - i\mathrm{Tr}\psi^\dagger \gamma^\mu D_\mu \psi \\ &\quad - V_{ferm} - V_{bos} \end{aligned} \tag{17}$$

  - V_{bos}: six scalar interaction term

$$V_{bos} \sim YY^\dagger YY^\dagger YY^\dagger \tag{18}$$

  - V_{ferm}: quartic mixed potentials

$$V_{ferm} \sim YY^\dagger \psi\psi^\dagger \tag{19}$$

- field content: transformation under the gauge group

$$Y \in (N, \bar{N}), \quad \psi \in (N, \bar{N}), \quad A \in (adj, 1) \quad \hat{A} \in (1, adj) \tag{20}$$

- perturbation theory: Chern-Simons level k occurs as an overall factor, so the coupling constant can be considered as

$$g_{CS}^2 = \frac{1}{k} \tag{21}$$

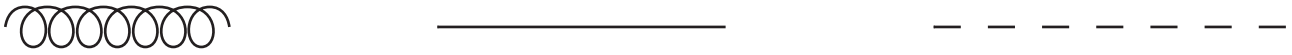
though k should be an integer to preserve the invariance under large scale gauge transformation. Also in the large N limit, using the double-line formalism, we see that each loop will provide an extra N factor, so the effective coupling constant is

$$\lambda \equiv g_{CS}^2 N = \frac{N}{k} \tag{22}$$

The theory become integrable in 't Hooft limit

$$k, N \to \infty, \qquad \lambda = \text{fixed} \tag{23}$$

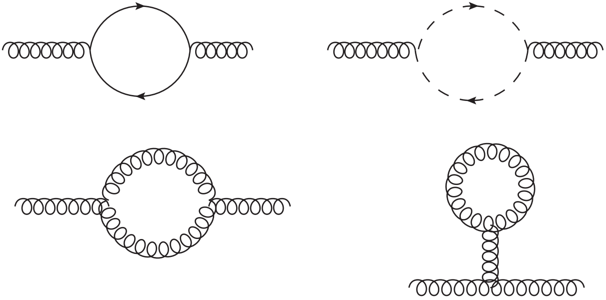
bare propagators:



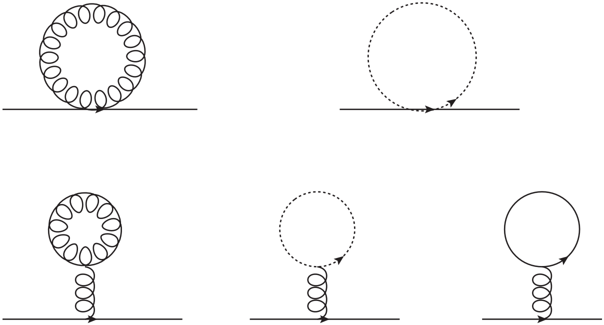
λ λ^0 λ^0

1-loop corrected propagtor:

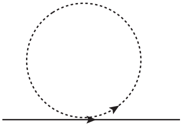
- gluon: λ^2 order

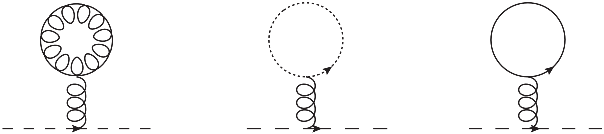


- scalar: λ order

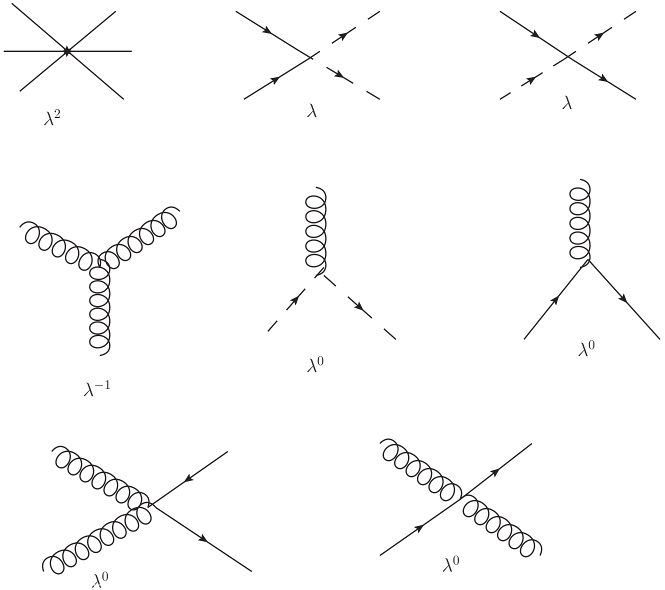


- fermion: λ order





- vertices:



λ^2
λ
λ
λ^{-1}
λ^0
λ^0
λ^0
λ^0

Integrability in ABJM

- As in SYM, the integrable structure is encoded in the ADM of the composite operators [Minahan&Zarembo, Bak&Rey].
- We consider the following single trace operators

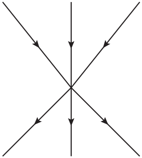
$$\hat{O} = \mathsf{Tr}\left(Y^{i_1} Y^{\dagger}_{j_1} \cdots Y^{i_L} Y^{\dagger}_{j_L}\right) \tag{24}$$

  which dual to the Hamiltonian of an alternating spin chain.
- To extract the ADM, we compute the two-point correlation functions

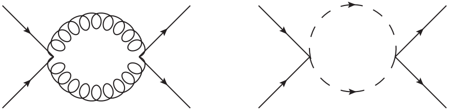
$$\langle \hat{O} \hat{O}^{\dagger} \rangle \tag{25}$$

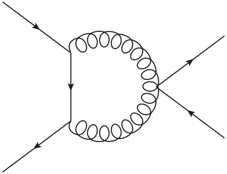
  by summing over all planar diagrams $(N \to \infty)$ in certain loop order λ

- at 2-loop order (λ^2), the contributions come from the following diagrams:
  - three-sites:

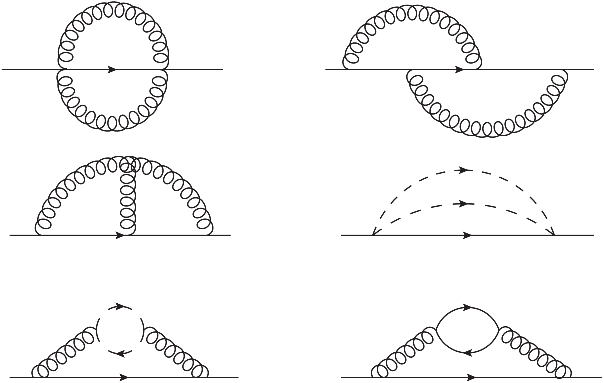


  - two-sites:





- one-site diagram:



The ADM at 2-loop order turns out to be

$$H_{2-loops} = \lambda^2 \sum_l \left[\mathbb{I} - \mathbb{P}_{l,l+2} + \frac{1}{2}\mathbb{P}_{l,l+2}\mathbb{K}_{l,l+1} + \frac{1}{2}\mathbb{P}_{l,l+2}\mathbb{K}_{l+1,l+2}\right] \tag{26}$$

where $\mathbb{K}$ is the trace operator, acting at two adjacent sites, defined in components as

$$\mathbb{K}^{i_1,j_2}_{j_1,i_2} = \delta^{i_1}_{i_2}\delta^{j_2}_{j_1} \tag{27}$$

The integrability is established by two kinds of R-matrices,

$$R^{\mathbf{44}}(u) = u + \mathbb{P}, \quad R^{\mathbf{4\bar{4}}}(u) = -(u+2) + \mathbb{K} \tag{28}$$

Supersymmetric Wilson loops in 3d CSM

- bosonic type: 1/6 BPS

$$W = \frac{1}{N} \mathsf{Tr}_{\mathcal{R}_1 \times \mathcal{R}_2} \exp \int \left(i A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| M_I^J Y^I Y_J^\dagger \right) ds \tag{29}$$

- fermionic type: 1/2 BPS

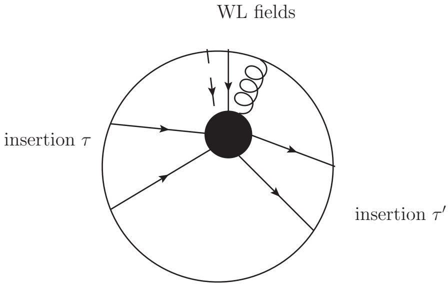
$$W = \frac{1}{N} \mathsf{Tr}_{\mathcal{R}} \exp \left(i \int L d\tau \right) \tag{30}$$

with the superconnection given by

$$L = \begin{pmatrix} i A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| M_I^J Y^I Y_J^\dagger, & \sqrt{\frac{2\pi}{k}} |\dot{x}| \eta^I \bar{\psi}_I \\ \sqrt{\frac{2\pi}{k}} |\dot{x}| \psi^I \bar{\eta}_I, & i \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| \hat{M}_J^I Y_I^\dagger Y^J \end{pmatrix} \tag{31}$$

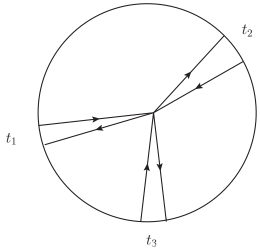
WL with Insertions

- For the 1/6 BPS WL, in order to get non-trivial boundary terms , we need at least two fields from each insertion to interact with the fields from WL,



Expand the WL to 1st order, we evaluate the expectation value below,

$$I = \langle Y(t_1)Y^\dagger(t_1)Y(t_3)Y^\dagger(t_3)\int dt_2 \frac{2\pi}{k} M_I^J Y^I(t_2)Y_J^\dagger(t_2)\rangle \tag{32}$$



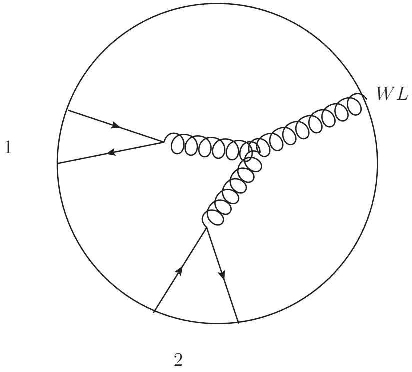
To fully contracted all the fields, we pick the sextic interaction vertex and roughly we get

$$\begin{aligned} I &\sim \lambda^3 \int dt_2 \int d^3w G^2(t_1 - w) G^2(t_2 - w) G^2(t_3 - w) \\ &\sim \lambda^3 \int dt_2 G^2(t_3 - t_1) \int d^3 w G^2(t_1 - w) G^2(t_2 - w) \\ &\sim \lambda^3 \int dt_2 G^2(t_3 - t_1) \int \frac{d^3 w}{|w|^4} \end{aligned} \tag{33}$$

- there is a linear divergence when $t_1 \sim t_2$ and $t_3 \sim t_2$
- this term is of order λ^3

For 1/2 BPS WL, at 2-loop order (λ^2), we give several diagrams which contribute to the bdy

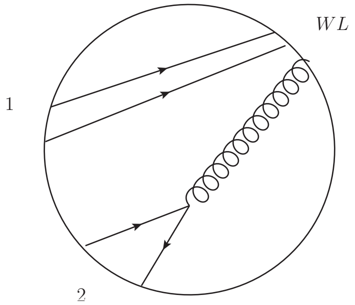
- Expand WL to 1st order



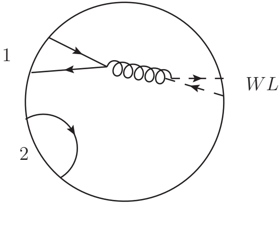
- expand WL to 2nd order, we will get the following new vertices from WL,

$$L^2 \sim \begin{pmatrix} A^2 + \lambda A Y Y^\dagger + \lambda^2 Y Y^\dagger Y Y^\dagger + \lambda \bar{\psi}\psi, & \lambda^{\frac{1}{2}} A \bar{\psi} + \lambda^{\frac{3}{2}} Y Y^\dagger \bar{\psi} \\ * & * \end{pmatrix} \quad (34)$$

some graphs at λ^2 order are



also two diagrams having fermions from WL



So we find that, at 2-loop order, boundary terms from the interactions of the WL and the insertions are of the type:

$$H_b = \alpha \mathbb{I} + \beta \mathbb{K} \tag{35}$$

The complete Hamiltonian is

$$H = \lambda^2 \sum_{l=1}^{2L-2} H_{l,l+1,l+2} + \beta \left(\mathbb{K}_{2L-1,2L} + \mathbb{K}_{1,2}\right) \tag{36}$$

This is an integrable Hamiltonian which can be seen most easily from CBA. For this, we make the following identifications:

$$Y^1 = A_1, \quad Y^2 = A_2, \quad Y^3 = B_1^\dagger, \quad Y^4 = B_2^\dagger \tag{37}$$

- vacuum:

$$|\Omega\rangle = |A_1 B_2 \cdots A_1 B_2\rangle \tag{38}$$

- elementary excitations: A type and B type

$$\begin{aligned} &|\cdots (A_2 B_2) \cdots\rangle \\ &|\cdots (B_1^\dagger B_2) \cdots\rangle \\ &|\cdots (A_1 A_2^\dagger) \cdots\rangle \\ &|\cdots (A_1 B_1) \cdots\rangle \end{aligned} \tag{39}$$

Under the action of trace operator, the state becomes

$$\mathbb{K}Y^aY_b^\dagger = \mathbb{K}_{i;b}^{a;j}Y^iY_j^\dagger = \delta_i^j\delta_b^aY^iY_j^\dagger = \delta_b^a\left(\sum_i Y^iY_i^\dagger\right) \tag{40}$$

so the Hamiltonian reduces to

$$H = \lambda^2\sum_{l=1}^{2L-2}\left(\mathbb{I} - \mathbb{P}_{l,l+2}\right) \tag{41}$$

There is no mixing of different excitations, so the spin wave for a single excitation X simply takes the form

$$\Psi_X(k) = \sum_{x=1}^{L}\left(e^{ikx} + R_X e^{-ikx}\right)|x\rangle \tag{42}$$

Solving the eigenvalue equation, we find

$$R_X = 1, \quad X = A_2, B_1^\dagger, A_2^\dagger, B_1 \tag{43}$$

So the reflection matrix is proportional to the identity, $R = \eta\mathbb{I}$. For an integrable theory, the reflection matrix should satisfy the REs,

$$\begin{aligned} S(k_1,k_2)R_{2l}(k_2)S(-k_2,k_1)R_{1l}(k_1) &= R_{1l}(k_1)S(-k_1,k_2)R_{2l}(k_2)S(-k_2,-k_1), \\ S(-k_1,-k_2)R_{1r}(-k_1)S(-k_2,k_1)R_{2r}(-k_2) &= R_{2r}(-k_2)S(-k_1,k_2)R_{1r}(-k_1)S(k_2,k_1). \end{aligned}$$

For our reflection matrix , the REs reduce to

$$S(k_1,k_2)S(-k_2,k_1) = S(-k_1,k_2)S(-k_2,-k_1) \tag{44}$$

We can check that the bulk S-matrix do satisfy the above relations.

Conclusion

- For $\mathcal{N} = 4$ SYM, the integrability is shown in $SU(2)$ sector. The integrability is also found in larger $SO(6)$ sector and in non-supersymmetric WL.
- For ABJM theory, it is possible to obtain non-trivial boundary terms from
  - higher loop order or larger closed sector
  - more complicated constructions of SUSY WLs
  - non-supersymmetric WLs

Acknowledgements

Thanks for your attentions!